Some Interpolation Problems Involving Interval Data

S. M. Markov*

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Abstract

Considered is interpolation of families of functions depending on a parameter by families of interpolation polynomials. Inner and outer inclusions for the interpolating families are constructed in terms of interval and extended interval arithmetic. Some interpolation polynomials involving dircted intervals are studied.

1 Introduction

Throughout the paper we consider interpolation involving algebraic polynomials. However, the results obtained can be easily generalised to comprise interpolation using other classes of interpolating functions, such as trigonometric polynomials, exponential functions etc.

Let $p_n(x, y; \xi)$ be the interpolation polynomial of degree $n-1$ taking at a given mesh $x = \{x_i\}_{i=1}^n \in X \subseteq R$, $x_1 < x_2 < ... < x_n$, prescribed values $y = \{y_i\}_{i=1}^n$. Using the Lagrange form we have

$$p_n(x, y; \xi) = \sum_{i=1}^n l_i(x) y_i, \quad l_i(\xi) = \prod_{j=1,\ j \neq i}^n ((\xi - x_j)/(x_i - x_j)). \tag{1}$$

Assume now that we are given intervals $Y_i = [y_i^-, y_i^+] \in I(R)$ for the values y_i. By $Y = \{Y_i\}_{i=1}^n$ we mean $y_i \in Y_i,\ i = 1, ..., n$. Consider the family of interpolation polynomials taking at x_i all possible values in the intervals Y_i, $i = 1, ..., n$:

$$p_n(x, Y; \xi) = \{p_n(x, y; \xi) \mid y \in Y\} = \{\sum_{i=1}^n l_i(\xi) y_i \mid y_i \in Y_i,\ i = 1, ..., n\}, \tag{2}$$

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where $l_i(\xi)$ are defined in (1). This family has been probably first investigated in [4]. For a fixed ξ the set $p_n(x, Y; \xi)$ is an interval. We shall denote the boundary functions of (2) by p_n^-, p_n^+, i. e. $p_n(x, Y; \xi) = [p_n^-(x, Y; \xi), p_n^+(x, Y; \xi)]$. It has been noticed [7], [8], [10] that (2) can be presented concisely by means of the interval-arithmetic operations for addition and for multiplication by real number defined for $[a^-, a^+], [b^-, b^+] \in I(R)$ by [1], [20], [21], [26]:

$$[a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+], \tag{3}$$

$$\alpha[a^-, a^+] = \{[\alpha a^{-\mathrm{sign}(\alpha)}, \alpha a^{\mathrm{sign}(\alpha)}],\ \alpha \neq 0;\ 0,\ \alpha = 0\}, \tag{4}$$

where for $\alpha \neq 0$, $\mathrm{sign}(\alpha) = \{-, \alpha < 0; +, \alpha > 0\}$. Using the interval arithmetic operations (3), (4) we can represent the set $\{\sum_{i=1}^n \alpha_i y_i \mid y_i \in Y_i\}$ as

$$\{\sum_{i=1}^n \alpha_i y_i \mid y_i \in Y_i, i = 1, ..., n\} = \sum_{i=1}^n \alpha_i Y_i. \tag{5}$$

Fixing ξ in (2) and applying relation (5) we obtain

$$p_n(x, Y; \xi) = \sum_{i=1}^n l_i(\xi) Y_i. \tag{6}$$

Formula (6) ofers a simple and remarkable example of a nontrivial application of interval arithmetic. Indeed, without interval arithmetic, the interval interpolation polynomial (6) can not be represented in a concise form. Using conventional techniques (see [4]) the interval-valued polynomial (6) can be described by its boundary functions which are piece-wise polynomial functions. More precisely, they are polynomials in each subinterval $[x_k, x_{k+1}]$ but generally in every two subintervals they are pieces of two different polynomials. The upper bound $p_n^+(x, Y)$ of (6) in the interval $[x_k, x_{k+1}]$ is (piece of) the interpolation polynomial $p_{n,k}^+$ passing through (determined by) the points (x_{k-2j}, y_{k-2j}^+), $(x_{k+2j+1}, y_{k+2j+1}^+)$, $(x_{k-2j-1}, y_{k-2j-1}^-)$, (x_{k+2j}, y_{k+2j}^-), $j = 0, 1, 2, ...$, where all mesh points are involved. The lower bound $p_n^-(x, Y)$ in the interval $[x_k, x_{k+1}]$ is (piece of) the interpolation polynomial $p_{n,k}^-$ passing through the alternative end-points of the vertical segments $(x_i, Y_i)_{i=1}^n$, that is the points having reversed $\pm$-signs as upper indeces for their y-components, namely (x_{k-2j}, y_{k-2j}^-), $(x_{k+2j+1}, y_{k+2j+1}^-)$, $(x_{k-2j-1}, y_{k-2j-1}^+)$, (x_{k+2j}, y_{k+2j}^+), $j = 0, 1, 2, ...$ However, this is valid only for the interval $[x_k, x_{k+1}]$. In another interval $[x_l, x_{l+1}]$ the boundary functions are (pieces of) other interpolation polynomials $p_{n,l}^-, p_{n,l}^+$ in general. Symbolically, without interval arithmetic we should write

$$p_n(x, Y; \xi) = [p_{n,k}^-(\xi), p_{n,k}^+(\xi)],\ \xi \in [x_k, x_{k+1}], \tag{7}$$

where $p_{n,k}^-, p_{n,k}^+$ are defined above. On the real line R we have

$$p_n(x, Y; \xi) = \bigcup_0^{n+1} [p_{n,k}^-(\xi), p_{n,k}^+(\xi)],$$

wherein $x_0 = -\infty, x_{n+1} = \infty$. According to (6) the set of all polynomials of $(n-1)$-st degree lying between p^- and p^+ coincides with the set (2)!

Note that the simple interval arithmetic expression (6) presents a complex interval function which boundaries are piece-wise polynomial functions. The "secret" is hidden in the fact that (6) actually comprises as many "normal" expressions as is the number of subintervals generated by the mesh points. Indeed, the signs of the Lagrangian coefficiens $l_i(\xi),\ i = 1, ..., n$, have particular values in each subinterval. This, as seen from (6) and (4), leads to particular expressions for the boundary functions in each subinterval.

Let us point out that the family of interpolation polynimials defined by (2) arbitrarily intersects the vertical segments $(x_i, Y_i), i = 1, ..., n$. It is practically important to study families which intersect (some of the) vertical segments in certain interdependent way. A practical situation which diserves interest is the situation when we know that (some of) the vertical segments are traced by the family monotonically in certain direction. To give an example consider a family of polynomials constructed by means of the above mentioned polynomials $p^-_{n,k}, p^+_{n,k}$ as follows

$$\{p_{n,k}(t) = (1-t)p^-_{n,k} + tp^+_{n,k} \mid t \in [0,1]\}. \tag{8}$$

The intersection points of this family with the vertical segments trace the vertical segments in an interdependent way. The vertical segments (x_k, Y_k), (x_{k+1}, Y_{k+1}) are traced in "positive direction", the next neighbouring segments (x_{k-1}, Y_{k-1}), (x_{k+2}, Y_{k+2}) are traced in "negative direction" and so on alternatively. The family (8) can be presented by a simple interval-arithmetic expression using directed intervals. However, we can not give analogous expression based on standard interval arithmetic.

Our approach will allow us to consider the alternative classical setting when interpolation is related not just to discrete numerical values but to a function f belonging to a given class. If f is sufficiently smooth, then for the distance between $y(t) = f(t; \cdot)$ and the corresponding interpolation polynomial $p_n(x, y(t))$ we have

$$|f(t;\xi) - p_n(x, y(t); \xi)| = (1/n!)|\partial f^{(n)}(t;\xi^0)/\partial\xi^n| \prod_{i=1}^{n} |\xi - x_i|, \tag{9}$$

where ξ^0 belongs to the interval comprising the points $\xi, x_1, x_2, ..., x_n$, symbolically $\xi^0 \in [\xi \vee x_1 \vee x_2 \vee ... \vee x_n]$ (see e. g. [24], [2]). We are interested in similar estimates in the situation when intervals are known for the values $f(x_i)$ (see [13] and [2] for a similar setting). In what follows we shall consider functions depending on one real-valued parameter.

2 Interpolation of families of functions depending on parameter

Let $f(t;\xi)$ be a real function defined on $T^* \bigotimes X^* \subseteq R^2$ which is continuous on $t \in T^* = [t_1^*, t_2^*] \in I(R)$. For every fixed $t \in T^*$, $f(t;\cdot)$ is a function defined on X^*, which we shall sometimes denote by $y(t) = f(t;\cdot)$. Denote the family of all $y(t)$ for $t \in T = [t^-, t^+] \subseteq T^*$ by

$$y(T) = f(T;\cdot) = \{f(t;\cdot) \mid t \in T\}. \tag{10}$$

For every $\xi \in X^*$ we have $f(T;\xi) \in I(R)$ so that (10) is an interval-valued function on X^*.

Fix $T \in I(R)$, $T \subseteq T^*$, and denote by $X_{f,T}$ the set of all $\xi \in X$, such that $f(t;\xi)$ is monotone w.r.t. t on T. If f is differentiable w.r.t. t, then for any fixed $\xi \in X_{f,T}$ the value of $\partial f(t;\xi)/\partial t$ does not change sign whenever t traces T. However, this sign may be different for two $\xi_1, \xi_2 \in X_{f,T}$, $\xi_1 \neq \xi_2$.

Let $y(t) \in y(T)$ and let $p_n(x, y(t); \xi)$ be the interpolation polynomial to $y(t)$ of degree $n-1$ along a given mesh $x = \{x_i\}_{i=1}^n \in X^*,\ x_1 < x_2 < ... < x_n$. Denote $y_i(t) = f(t; x_i),\ i = 1, ..., n$. Using the Lagrange form (1) we have

$$p_n(x, y(t); \xi) = \sum_{i=1}^n l_i(\xi) y_i(t), \qquad l_i(\xi) = \prod_{j=1,\ j\neq i}^n ((\xi - x_j)/(x_i - x_j)). \tag{11}$$

Denote the range of $f(t;x_i)$ over T by $y_i(T) = f(T;x_i) = \{f(t;x_i) \mid t \in T\}$. Assume that the family of interpolation polynomials (6) has been generated by the values $y_i(T)$ of the interval-valued function (10) at the mesh points x_i and consider the distance between both interval-valued functions at points different from the mesh points x. Note that $p_n(x, Y; \cdot)$ with $Y = \{y_i(T)\}_{i=1}^n$, as defined by (2), (6), may include polynomials $p_n(x, y; \xi) = \sum_{i=1}^n l_i(\xi) y_i,\ y_i \in y_i(T),\ i = 1, ..., n$, which do not interpolate any individual function $f(t;\cdot)$ from the family $\{f(t;\cdot) \mid t \in T\}$ unless all $y_i(T)$ are degenerate (point-wise) intervals (we have $y \neq y(t) = f(t;x)$, in general). Therefore for the distance between the intervals $p_n(x, Y; \xi)$ and $f(T;\xi)$ at $\xi \neq x_i$ we can not use estimates in terms of smoothness of f similar to (9) which are valid for the degenerate case $T = t \in T^*$. The following theorem deals with an example of a family $y(T)$ of the type (10), which can be approximated in a certain interval by the interval valued polynomial (6) and the distance between the family $y(T)$ and the interval polynomial can be estimated in terms of the smoothness of y. As a measure for the distance between two intervals $A, B \in I(R)$ we take $r(A, B) = r([a^-, a^+], [b^-, b^+]) = \max\{|a^- - b^-|, |a^+ - b^+|\}$. We also make use of $|A| = \max\{|a^-|, |a^+|\}$.

Theorem 1. Let $f(t;\xi)$ be monotone increasing (decreasing) w.r.t. $t \in T$ at the mesh points $x_{k-2j}, x_{k+2j+1}, j = 0, 1, 2, ...$ and monotone decreasing (increasing) at $x_{k-2j-1}, x_{k+2j},\ j = 0, 1, 2, ...$ (all mesh points are involved in alternating order starting from the points x_k, x_{k+1} towards outside). Denote $y(t) = f(t;\cdot)$

and the corresponding interpolation polynomial by $p_n(x, y(t); \xi)$. The following properties take place:

i) $p_n(x, y(t); \xi)$ is monotone increasing (decreasing) in t for $\xi \in [x_k, x_{k+1}]$;

ii) $\{p_n(x, y(t); \xi) \mid t \in T\} = p_n(x, y(T); \xi) = \sum_{i=1}^{n} l_i(\xi) f(T; x_i)$ for $\xi \in [x_k, x_{k+1}]$;

iii) if f is n times differentiable w. r. t. ξ, then $r(p_n(x, y(T); \xi), f(T; \xi)) \leq (1/n!)|\partial f^{(n)}(T; X)/\partial \xi^n| \prod_{i=1}^{n} |\xi - x_i|$ for $\xi \in [x_k, x_{k+1}]$, where $X = [\xi \vee x_1 \vee x_2 \vee ... \vee x_n]$, i. e. X is the smallest interval comprising the mesh points $\{x_i\}_{i=1}^{n}$ and ξ.

Proof. From $p(\xi) = p_n(x, y(t); \xi) = \sum_{i=1}^{n} l_i(\xi) f(t; x_i)$ we have

$$dp^n(\xi)/dt = \sum_{i=1}^{n} l_i(\xi) \partial f(t; x_i)/\partial t.$$

Let $x_k \leq \xi \leq x_{k+1}$. Then the polynomials of $(n-1)$-st degree $l_{k-2j}(\xi)$, $l_{k+2j+1}(\xi)$, $j = 0, 1, 2, ...$, are positive for $\xi \in [x_k, x_{k+1}]$ whereas the polynomials $l_{k-2j}(\xi)$, $l_{k+2j+1}(\xi)$, $j = 0, 1, 2, ...$, are negative in $[x_k, x_{k+1}]$. The assumption of the theorem says that in $[x_k, x_{k+1}]$, we have $\mathrm{sign}(\partial f(t; x_i)/\partial t) = \mathrm{sign} l_i(\xi)$ $(= -\mathrm{sign} l_i(\xi))$, $i = 1, 2, ...$, and hence $dp_n(\xi)/dt > 0$ (< 0) for $x_k \leq \xi \leq x_{k+1}$, that is case i) is proved.

To show ii) we have to observe that for $\xi \in [x_k, x_{k+1}]$ the boundary functions of the interval polynomial $p_n(x, Y; \xi)$ and of the set $\{p_n(x, y(t); \xi) \mid t \in T\}$ are polynomials of $(n-1)$-degree which have same values at the mesh points $x = \{x_i\}_{i=1}^{n}$ and therefore coincide. However, note that the boundary functions of these sets may not coincide outside the interval $[x_k, x_{k+1}]$ where they are pieces of other polynomials. Actually the boundary functions of the family $\{p_n(x, y(t); \xi) \mid t \in T\}$ are $p_{n,k}^{-}, (\xi), p_{n,k}^{+}(\xi)]$ for all ξ (see (7)).

To demonstrate iii) note that for $t = t^-, t^+$ we have

$$\begin{aligned} |f(t; \xi) - p_n(x, y(t); \xi)| &= (1/n!)|\partial f^{(n)}(t; \xi^0)/\partial \xi^n| \prod_{i=1}^{n} |\xi - x_i| \\ &\leq (1/n!)|\partial f^{(n)}(t; X)/\partial \xi^n| \prod_{i=1}^{n} |\xi - x_i|, \end{aligned}$$

where $\xi^0 \in X = [\xi \vee x_1 \vee x_2 \vee ... \vee x_n]$. Variation of this inequality for $t \in T$ and minding the monotonicity of f and p_n implies the validity of iii) in $[x_k, x_{k+1}]$.

Case iii) of Theorem 1 shows that for the special choice of the family $f(T; \xi)$ described in the theorem, we are able to give an estimate for the distance between $f(T, \xi)$ and the corresponding family of interpolating polynomial functions $\{p_n(x, f(t, \cdot)) \mid t \in T\}$ in $[x_k, x_{k+1}]$.

Theorem 1 ii) says that for the family $y(t) = f(t; \cdot)$ considered in the theorem the corresponding family of interval polynomials for $\xi \in [x_k, x_{k+1}]$ is given by the simple interval-arithmetic expression

$$\{p_n(x, f(t;x);\xi) \mid t \in T\} = \sum_{i=1}^{n} l_i(\xi) f(T; x_i).$$

However, the above relation is true only for $\xi \in [x^k, x^{k+1}]$ and for the very restrictive case considered in the theorem. In what follows we shall give similar interval-arithmetic expressions under the more general assumption that $f(t;\xi)$ is monotone on t at each mesh point $\xi = x_i$, without having to specify the kind of monotonicity at x_i as this was required in Theorem 1. To this end we shall make use of extended interval arithmetic. The results can be equally well formulated either by using normal intervals and nonstandard operations [14]–[16], [18], or by using directed intervals [9], [11], [12], [19], [22]. In what follows we shall make use of the latter form. To this end we next give some basic concepts of the extended interval arithmetic using directed intervals.

3 An interpolation polynomial involving directed intervals

A directed interval on R is a pair of reals $[a^-, a^+]$, $a^-, a^+ \in R$. The set of all directed intervals is denoted by D. Addition of directed intervals and multiplication by a real number $\alpha \in R$ are defined as extensions of (3), that is:

$$[a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+],\ [a^-, a^+], [b^-, b^+] \in D; \quad (12)$$
$$\alpha[a^-, a^+] = [\alpha a^{-\mathrm{sign}(\alpha)}, \alpha a^{\mathrm{sign}(\alpha)}],\ [a^-, a^+] \in D, \alpha \neq 0; \quad 0[a^-, a^+] = 0. \quad (13)$$

Whenever appropriate, we shall denote directed intervals by boldface letters. The basic operations (12), (13) involve a variety of derivative operations. We define *negation* by $-\mathbf{A} = (-1)\mathbf{A} = [-a^+, -a^-]$, resp. *subtraction* by $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = [a^- - b^+, a^+ - b^-]$. To every $\mathbf{A}$ there exists *additive inverse* directed interval $-_h\mathbf{A} = [-a^-, -a^+]$, generating the operation *hyperbolic subtraction* $\mathbf{A} -_h \mathbf{B} = \mathbf{A} + (-_h\mathbf{B}) = [a^- - b^-, a^+ - b^+]$. The *conjugated* (*dual*) directed interval is defined by $\mathbf{A}_- = -(-_h\mathbf{A}) = -_h(-\mathbf{A}) = [a^+, a^-]$. Note that $\mathbf{A} -_h \mathbf{B} = \mathbf{A} - \mathbf{B}_-$, which is $\neq \mathbf{A} - \mathbf{B}$ in general. The *direction* of $\mathbf{A}$ is defined by $\tau(\mathbf{A}) = +$, if $a^- \leq a^+$ and $\tau(\mathbf{A}) = -$, otherwise. Directed intervals with positive direction are called positively didected (not to be confused with positive!) or *proper* intervals. Denote $\mathbf{A}_+ = \mathbf{A}$. Then the directed interval $\mathbf{A}_{\tau(\mathbf{A})}$ has a positive direction for every $\mathbf{A} \in D$ and is called the *proper part* (or the *prop*) of $\mathbf{A}$, symbolically $\mathrm{prop}(\mathbf{A}) = \mathbf{A}_{\tau(\mathbf{A})}$. The set of all proper intervals is equivalent to the set of normal intervals $I(R)$ and will be denoted by $I(R)$. The set of negatively directed (improper) intervals is denoted by $I(R)_-$. The set D with the operations (12), (13) satisfies all basic relations of a linear space exept for the relation $(\alpha + \beta)\mathbf{C} = \alpha\mathbf{C} + \beta\mathbf{C}$. This relation is replaced in D by $(\alpha + \beta)\mathbf{C}_{\mathrm{sign}(\alpha+\beta)} = \alpha\mathbf{C}_{\mathrm{sign}(\alpha)} + \beta\mathbf{C}_{\mathrm{sign}(\beta)}$.

Since D is an extension of $I(R)$ we shall assume that all relations in $I(R)$ hold true also in the set of proper intervals. For a proper interval $\mathbf{A} \in D$ we write $A = \mathbf{A} = \text{prop}(\mathbf{A}) \in I(R)$. In particular, inclusion between proper intervals is well defined in the usual manner. We define inclusion between directed intervals via inclusion between their corresponding props by setting $\mathbf{A} \subseteq \mathbf{B} \Longleftrightarrow \text{prop}(\mathbf{A}) \subseteq \text{prop}(\mathbf{B})$ for any two $\mathbf{A},\ \mathbf{B} \in D$ such that $\mathbf{B} \neq \mathbf{A}_-$. For two dual intervals $\mathbf{A},\ \mathbf{B} = \mathbf{A}_-$ we may postulate that the negatively directed interval is included in the positively directed one. If an expression involves both directed and normal intervals, we shall consider the normal intervals as proper directed intervals. If an interval expression involves at least one directed interval or at least one (purely) directed operation or relation (such as conjugation) then this expression will be considered as expression between directed intervals. In accordance to these stipulations inclusion between normal and directed intervals also make sense, namely, $A \subseteq \mathbf{B} \Longleftrightarrow A \subseteq \text{prop}(\mathbf{B})$, resp. $\mathbf{A} \subseteq B \Longleftrightarrow \text{prop}(\mathbf{A}) \subseteq B$. For $a \in R$, $\mathbf{A} \in D$, the inclusion $a \in \mathbf{A}$ is equivalent to $a \in A$ or $a \in \text{prop}\mathbf{A}$. We note that this definition of inclusion slightly differs from the one considered by E. Kaucher [12]. The distance between two directed intervals is defined as $r(\mathbf{A}, \mathbf{B}) = |\mathbf{A} -_h \mathbf{B}|$, wherein $|\mathbf{A}| = \max\{|a^-|, |a^+|\}$. The width is defined by $\omega(\mathbf{A}) = \omega(\text{prop}(\mathbf{A})) = |a^+ - a^-|$.

We next give two propositions involving expressions for the sum of directed intervals in terms of the set theoretic operations for (joint) union $\bigcup$ and intersection $\bigcap$. The union and the intersection of two equaly directed intervals are directed intervals having the direction of the arguments involved and their props are defined by $\text{prop}(\mathbf{A} \bigcup \mathbf{B}) = \text{prop}(\mathbf{A}) \bigcup \text{prop}(\mathbf{B})$. $\text{prop}(\mathbf{A} \bigcap \mathbf{B}) = \text{prop}(\mathbf{A}) \bigcap \text{prop}(\mathbf{B})$, resp.

Proposition 1. i) For $\mathbf{A},\ \mathbf{B} \in D$ such that $\tau(\mathbf{A}) \neq \tau(\mathbf{B})$ we have

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{cases} \bigcap_{a \in A}(a + \mathbf{B}), & \text{if } \omega(\mathbf{A}) \leq \omega(\mathbf{B}), \\ \bigcap_{b \in B}(\mathbf{A} + b), & \text{if } \omega(\mathbf{A}) \geq \omega(\mathbf{B}); \end{cases} \\ &= \bigcap_{a \in A}(a + \mathbf{B}) \bigcup \bigcap_{b \in B}(\mathbf{A} + b). \end{aligned} \tag{14}$$

ii) For $\mathbf{A},\ \mathbf{B} \in D$ such that $\tau(\mathbf{A}) = \tau(\mathbf{B})$

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \bigcup_{a \in A}(a + \mathbf{B}) = \bigcup_{b \in B}(\mathbf{A} + b) \\ &= \bigcup_{a \in A}(a + \mathbf{B}) \bigcap \bigcup_{b \in B}(\mathbf{A} + b). \end{aligned} \tag{15}$$

We note that the union (14) involves an empty interval; the intersection (15) involves two equally directed intervals. Formulae (14) and (15) express the duality between the expressions for $\mathbf{A} + \mathbf{B}$ in both cases, which is equivalent to considering both expressions $\mathbf{A} + \mathbf{B}$, $\mathbf{A} + \mathbf{B}_-$. To clarify this we present Proposition 1, i) in the following eqiuvalent form

Proposition 1$'$. For $\mathbf{A},\ \mathbf{B} \in I(R)$ we have

$$\mathbf{A} + \mathbf{B}_- = \begin{cases} \bigcap_{b\in B}(\mathbf{A} + b), & \text{if } \mathbf{A} + \mathbf{B}_- \in I(R), \\ \bigcap_{a\in A}(a + \mathbf{B}), & \text{if } \mathbf{A}_- + \mathbf{B} \in I(R)_-. \end{cases}$$

The next proposition deals with a sum of n directed intervals and is a generalisation of Proposition 1. We first introduce some notations. Let $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n) \in D^n$ be a vector of directed intervals. If all components $\mathbf{A}_i$, $i = 1, ..., n$, of $\mathbf{A}$ have same direction $\tau(\mathbf{A}_i)$, then the direction of the vector $\mathbf{A}$ is defined by $\tau(\mathbf{A}) = \tau(\mathbf{A}_i)$. For a real vector $a = (a_1, a_2, ..., a_n) \in R^n$ the inclusion $a \in \mathbf{A}$ means that $a_i \in \mathbf{A}_i$, $i = 1, ..., n$. Denote further $\Sigma(\mathbf{A}) = \sum_{i=1}^n \mathbf{A}_i$, and, in particular $\Sigma(a) = \sum_{i=1}^n a_i$.

Let $\mathbf{A}' = (\mathbf{A}_{i_1}, \mathbf{A}_{i_2}, ..., \mathbf{A}_{i_k}) \in D^k$, $\mathbf{A}'' = (\mathbf{A}_{i_{k+1}}, \mathbf{A}_{i_{k+2}}, ..., \mathbf{A}_{i_n}) \in D^{n-k}$ be two subsets of the interval vector $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n)$, such that $\mathbf{A} = \{\mathbf{A}', \mathbf{A}''\}$, $1 \le k \le n$. The couple $(\mathbf{A}', \mathbf{A}'')$ will be called a partition of $\mathbf{A}$.

Proposition 2. Let $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n) \in D^n$ and let $(\mathbf{A}', \mathbf{A}'')$ be a partition of $\mathbf{A}$.

i) If all components of $\mathbf{A}$ are of same direction and $(\mathbf{A}', \mathbf{A}'')$ is an arbitrary partition of $\mathbf{A}$, then

$$\begin{aligned} \sum_{i=1}^n \mathbf{A}_i &= \mathbf{A}_1 + \mathbf{A}_2 + ... + \mathbf{A}_n = \Sigma(\mathbf{A}') + \Sigma(\mathbf{A}'') \\ &= \bigcup_{a'\in A'} \Sigma(a') + \Sigma(\mathbf{A}'') = \bigcup_{a''\in A''} \Sigma(\mathbf{A}') + \Sigma(a''). \end{aligned}$$

ii) If the components of $\mathbf{A}$ are of different directions and $(\mathbf{A}', \mathbf{A}'')$ is a partition of $\mathbf{A}$, such that $\mathbf{A}'$, resp. $\mathbf{A}''$, comprise intervals of same direction, then

$$\begin{aligned} \sum_{i=1}^n \mathbf{A}_i &= \mathbf{A}_1 + \mathbf{A}_2 + ... + \mathbf{A}_n = \Sigma(\mathbf{A}') + \Sigma(\mathbf{A}'') \\ &= \begin{cases} \bigcap_{\Sigma(a')\in\Sigma(A')} \Sigma(a') + \Sigma(\mathbf{A}''), & \text{if } \omega(\Sigma(\mathbf{A}')) \le \omega(\Sigma(\mathbf{A}'')) \\ \bigcap_{\Sigma(a'')\in\Sigma(A'')} \Sigma(\mathbf{A}') + \Sigma(a''), & \text{if } \omega(\Sigma(\mathbf{A}')) \ge \omega(\Sigma(\mathbf{A}'')) \end{cases} \\ &= \begin{cases} \bigcap_{a'\in A'} \Sigma(a') + \Sigma(\mathbf{A}''), & \text{if } \omega(\Sigma(\mathbf{A}')) \le \omega(\Sigma(\mathbf{A}'')) \\ \bigcap_{a''\in A''} \Sigma(\mathbf{A}') + \Sigma(a''), & \text{if } \omega(\Sigma(\mathbf{A}')) \ge \omega(\Sigma(\mathbf{A}'')) \end{cases} \\ &= \bigcap_{a'\in A'} (\Sigma(a') + \Sigma(\mathbf{A}'')) \bigcup \bigcap_{a''\in A''} (\Sigma(\mathbf{A}') + \Sigma(a'')). \end{aligned}$$

Let us consider now an interpolation polynomial of the form (6) involving directed intervals. Let $x = \{x_i\}_{i=1}^n \in X$ be a mesh and $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n) \in D^n$ be a vector of directed intervals. Consider the expression

$$p_n(x, \mathbf{Y}; \xi) = \sum_{i=1}^n l_i(\xi)\mathbf{Y}_i. \tag{16}$$

For every fixed ξ the value of (16) is a directed interval, which can be computed by means of (12), (13). At the mesh points the values of (16) are the given directed intervals $\mathbf{Y}_i$.

We next give a set-theoretic interpretation for the value of (16) at arbitrary point ξ. If all $\mathbf{Y}_i$, $i = 1, ..., n$, have same direction, then, according to Proposition 2 i), the directed interval $p_n(x, \mathbf{Y}; \xi)$ has same direction and proper part $\mathrm{prop}(p_n(x, \mathbf{Y}; \xi)) = p_n(x, Y; \xi)$ as defined by (6). If $\mathbf{Y}_i$ have different directions, let us consider a partition $(\mathbf{Y}', \mathbf{Y}'')$ of $\mathbf{Y}$, such that $\mathbf{Y}'$, resp. $\mathbf{Y}''$, consist of equally directed intervals and $\tau(\mathbf{Y}') \neq \tau(\mathbf{Y}'')$. To be more specific we shall assume w. l. g. that $\tau(\mathbf{Y}') = -$, $\tau(\mathbf{Y}'') = +$ and $\mathbf{Y}' = (\mathbf{Y}_{i_1}, \mathbf{Y}_{i_2}, ..., \mathbf{Y}_{i_k}) \in D^k$, $\mathbf{Y}'' = (\mathbf{Y}_{i_{k+1}}, \mathbf{Y}_{i_{k+2}}, ..., \mathbf{Y}_{i_n}) \in D^{n-k}$ for some k, $1 \leq k \leq n$. We shall further denote by $y' \in R^k$, $y'' \in R^{n-k}$ to real vectors, such that $y' \in \mathbf{Y}'$, resp. $y'' \in \mathbf{Y}''$ (we write below $y' \in Y'$, resp. $y'' \in Y''$, which is the same). Using Proposition 2 ii) we obtain

$$\begin{aligned}
p_n(x, \mathbf{Y}; \xi) &= \sum_{i=1}^{n} l_i(\xi)\mathbf{Y}_i = \sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j} \\
&= \bigcap_{y' \in Y'} (\sum_{j=1}^{k} l_{i_j}(\xi) y_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j}) \\
&\bigcup \bigcap_{y'' \in Y''} (\sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi) y_{i_j}).
\end{aligned}$$

The first intersection $\bigcap_{y' \in Y'}(\sum_{j=1}^{k} l_{i_j}(\xi) y_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j})$ in the above expression involves only positively directed (proper) intervals, whereas the second intersection $\bigcap_{y'' \in Y''}(\sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j} + \sum_{j=k+1}^{n} l_{i_j}(\xi) y_{i_j})$ involves only negatively directed (improper) intervals. One of the intersections is empty unless both intersections produce as results same real values, that is degenerate intervals. Namely, from Proposition 2 ii), if $\omega(\sum_{j=1}^{k} l_{i_j}(\xi)\mathbf{Y}_{i_j}) < \omega(\sum_{j=k+1}^{n} l_{i_j}(\xi)\mathbf{Y}_{i_j})$, then the second intersection is empty, if the oposite inequality holds, then the first intersection is empty, if an equality takes place, then both intersections have equal real values. This shows that the boundary functions of (16) are piece-wise polynomial functions.

Polynomials of the form (16) find application in the computation of L_k-compatible systems of interval segments as introduced in [17].

4 Interpolation of parametric families using directed ranges

Let $f(t)$ be continuous and monotone function on $T = [t^-, t^+] \in I(R)$, which will be denoted by $f \in CM(T)$. The directed interval $[f(t^-), f(t^+)]$ is called the *directed range* of f and will be denoted by $f[T]$ or $\mathbf{f}[T]$. Clearly, the directed range $\mathbf{f}[T]$ comprises information about: i) the range $f(t)$ and ii) the kind of monotonicity of f on T (nondecreasing/nonincreasing). We give some simple rules for computing with ranges and directed ranges of monotone functions (see also [25]). We denote for brevity $\tau(f[T]) = \tau_f$.

Rule 1. If $f, g \in CM(T)$, then $\mathbf{f}[T] + \mathbf{g}[T] \subseteq (f+g)(T) \subseteq f(T) + g(T)$. For the width of $f(T)+g(T)$ we have $\omega_1 = \omega(\mathbf{f}[T] + \mathbf{g}[T]) \le \omega((f+g)(T)) \le \omega(f(T)+g(T)) = \omega_2$. The upper bound $\omega((f+g)(T))$ can be improved by $\omega((f+g)(T)) \le (\omega_1+\omega_2)/2$.

Rule 2 [18]. If, in addition to the assumption of Rule 1: $f, g \in CM(T)$, we assume $h = f+g \in CM(T)$, then $\mathbf{h}[T] = (f+g)[T] = \mathbf{f}[T] + \mathbf{g}[T]$.

Rule 3 [18]. If f is monotone on $T \in I(R)$ and $\alpha \in R$, then for $h = \alpha f$ we have $\mathbf{h}[T] = \alpha \mathbf{f}[T]_{\mathrm{sign}(\alpha)}$.

Rules 1 − 3 imply the following

Rule 4. If: i) $f_i \in CM(T)$, $i = 1, ..., n$, and ii) $\alpha_i \in R$, $i = 1, ..., n$, then

$$\sum_{i=1}^{n} \alpha_i \mathbf{f}_i[T]_{\mathrm{sign}(\alpha_i)} \subseteq (\sum_{i=1}^{n} \alpha_i f_i)(T) \subseteq \sum_{i=1}^{n} \alpha_i f_i(T). \tag{17}$$

Rule 5. If, in Rule 4 in addition to i) − ii) we assume: iii) $h = \sum_{i=1}^{n} \alpha_i f_i \in CM(T)$, then

$$\mathbf{h}[T] = (\sum_{i=1}^{n} \alpha_i f_i)[T] = \sum_{i=1}^{n} \alpha_i \mathbf{f}_i[T]_{\mathrm{sign}(\alpha_i)}. \tag{18}$$

Remarks. 1. Note that Rule 2 does not presume monotonicity of $f+g$ and that $\mathbf{f}[T]+\mathbf{g}[T]$ gives substantially inner bounds for $(f+g)(T)$ if f, g are differently monotone. For equally monotone functions f, g the sum is also monotone and we can apply Rule 1. However, Rule 2 is valid also for equally monotone functions, in this case $\mathrm{prop}(\mathbf{f}[T]+\mathbf{g}[T]) = f(T)+g(T)$. Rule 2 can be also expressed in one of the following way: i) If $f, g \in CM(T)$, then $(f+g)(T)$ lies between (w. r. t. $\subseteq$) $\mathbf{f}[T]+\mathbf{g}[T]$ and $\mathbf{f}[T]+\mathbf{g}[T]_{\tau_f \tau_g}$; ii) $f(T)+g(T)_{\tau_f \tau_g} \le (f+g)(T) \le f(T)+g(T)$. 2. In Rule 3 the lower index $\mathrm{sign}(\alpha_{\mathrm{i}})$ changes the direction of the directed range $\mathbf{f}$[T] according to the sign of α_i. Note that the multiplication by real number does not change the direction of the directed interval. 3. Rules 1 and 3 obtain a simple form for linear functions f, g, resp. f_i, since then the sums $f+g$, resp. $\sum f_i$ are also linear and therefore monotone. 4. The above rules can be successfully incorporated in an algorithm which automatically finds ranges of functions and their derivatives, such as the one reported in [3], and an extended interval differentiation arithmetic can be developed in the sense of [23].

We next apply the arithmetic for directed intervals to interval-valued functions corresponding to parametric families of functions. It is interesting to note that interval-valued functions generated by parametric families have been considered in an early paper on interval arithmetic [26].

Assume that $f(t;\xi)$ is continuous on $T^* \bigotimes X^*$ and that for every ξ belonging to some nonempty set $X_{f,T} \subseteq X^*$, $f(t;\xi) \in CM(T)$, $T = [t^-, t^+] \in I(R)$, $T \subseteq T^*$. In addition to the interval-valued function $f(T;\cdot) = \{f(t;\cdot) \mid t \in T\}$ defined on X^* we can consider a mapping $\mathbf{f}[T;\cdot] : X_{f,T} \longrightarrow D$ defined for $\xi \in X_{f,T}$ by $\mathbf{f}[T;\xi] = [f(t^-;\xi), f(t^+;\xi)]$, which is the directed range of $f(t;\xi)$ over T.

Let $x_i \in X_{f,T}$, $i = 1, ..., n$, $x_1 < x_2 < ...x_n$, that is the functions $f(t;x_i)$, $i = 1, ..., n$, are monotone on T. Then the directed ranges $\mathbf{f}[T;x_i]$ are defined by $\mathbf{f}[T;x_i] = [f(t^-;x_i), f(t^+;x_i)]$. Each $f(t;\cdot)$ generates an interpolation polynomial p_n passing through the points $(x, f(t;x)) = (x_i, f(t;x_i))_{i=1}^n$:

$$p_n(x, f(t;x);\xi) = \sum_{i=1}^{n} l_i(\xi) f(t;x_i). \tag{19}$$

Theorem 2. Assume that the function $f(t;\xi)$ is continuous on $T^* \bigotimes X^*$ and the functions $f(t;x_i)$, $i = 1, ..., n$, are monotone on $T \in T^*$. Then: i) for every ξinR

$$\sum_{i=1}^{n} l_i(\xi)\mathbf{f}[T;x_i]_{\mathrm{sign}(l_i(\xi))} \subseteq p_n(x, f(T;x);\xi) \subseteq \sum_{i=1}^{n} l_i(\xi) f(T;x_i). \tag{20}$$

ii) if (19) is monotone on T at $\xi \in R$, then $p_n(x, f(T;x);\xi)$ reaches its lower bound in (20, i. e.

$$p_n(x, f(T;x);\xi) = \sum_{i=1}^{n} l_i(\xi)\mathbf{f}[T;x_i]_{\mathrm{sign}(l_i(\xi))} \tag{21}$$

and, if f is n times differentiable w. r. t. ξ, then $r(p_n(x, y(T);\xi), f(T;\xi)) \leq (1/n!)|\partial f^{(n)}(T;X)/\partial \xi^n| \prod_{i=1}^n |\xi - x_i|$ for $\xi \in [x_k, x_{k+1}]$, where X is an interval comprising the mesh points $\{x_i\}_{i=1}^n$ and ξ.

Proof. The proof follows by fixing ξ and applying Rules 4 and 5. Equality (21) is obvious from the more detailed form

$$\begin{aligned} \mathbf{p}_n(x, \mathbf{f}[T;x];\xi) &= [p_n(x, f(t^-;x);\xi), p_n(x, f(t^+;x);\xi)] \\ &= \sum_{i=1}^{n} l_i(\xi)\mathbf{f}[T;x_i]_{\mathrm{sign}(l_i(\xi))}. \end{aligned}$$

An open problem is to find estimates for the interpolation family in the situation when the family f is monotone at some of the knotes x_i (and not at all of them).

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Address of Author
Institute of Biophysics
Bulgarian Academy of Sciences
Acad. G. Bonchev st., blng. 21
1113 Sofia, BULGARIA